# An Antiderivative

# of $x^{\alpha-1} e^{\beta x}$

# $(\alpha, \beta > 0)$

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# 1. Tools

**Def.:** Let $J$ be a non-empty interval of $\mathbb{R}$.
Let $\phi : J \to \mathbb{R}$ be a mapping.
We now define:

1. $\phi : J \to \mathbb{R}$ is convex, iff
   $\forall x, y \in J \quad \forall t \in [0; 1] \quad \phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y)$

2. Let $\phi(J) \subseteq \mathbb{R}_+$.
   $\phi : J \to \mathbb{R}$ is logarithmically convex, iff
   $\ln(\phi) : J \to \mathbb{R}$ is convex

**Rem.:** Let $\phi(J) \subseteq \mathbb{R}_+$.
Because $\exp : \mathbb{R} \to \mathbb{R}$ is convex and monotonically increasing, we get the following:

$(\phi : J \to \mathbb{R}$ is logarithmically convex$) \Rightarrow$
$(\phi : J \to \mathbb{R}$ is convex$)$

**Theo.:**

**Pre.:** Let $J$ be a non-empty interval of $\mathbb{R}$.
Let $\phi : J \to \mathbb{R}$ be a differentiable mapping.

**Ass.:** $(\phi : J \to \mathbb{R}$ is convex$) \Leftrightarrow$
$(\phi' : J \to \mathbb{R}$ is monotonically increasing$)$

**Theo.:**

**Pre.:** Let $J$ be a non-empty interval of $\mathbb{R}$.
Let $\phi : J \to \mathbb{R}$ be a 2-times differentiable mapping.

**Ass.:** $(\phi : J \to \mathbb{R}$ is convex$) \Leftrightarrow$
$\phi'' \geq 0$

# 2. Gamma-Function

The Gamma-Funktion $\Gamma : \mathbb{R}_+ \to \mathbb{R}$ is for all $\alpha \in \mathbb{R}_+$ defined through the absolutely convergent integral

$$\Gamma(\alpha) := \underbrace{\int_0^\infty \tau^{\alpha-1} \cdot e^{-\tau} \, d\tau}_{>0}$$

From literature we have:

$$\Gamma : \mathbb{R}_+ \to \mathbb{R} \text{ is analytically} \tag{1}$$

$$\forall \alpha \in \mathbb{R}_+ \quad \Gamma(\alpha + 1) = \alpha \cdot \Gamma(\alpha) \tag{2}$$

$$\forall k \in \mathbb{N}_0 \quad \Gamma(k + 1) = k! \tag{3}$$

$$\Gamma : \mathbb{R}_+ \to \mathbb{R} \text{ is logarithmically convex (and ergo convex)} \tag{4}$$

$$\Gamma(1) = 1 \text{ and } \Gamma(2) = 1 \tag{5}$$

With (4) and (5) we have:

$$\Gamma \mid [2; \infty[ \text{ is monotonically increasing} \tag{6}$$

# 3. Idea

Let $x = \left(\mathrm{id}_{\mathbb{R}}\right) \mid \mathbb{R}_+$.
We now define a mapping $\gamma : ]-1;\infty[ \; \to \mathbb{R}$ through

$$\forall u \in ]-1;\infty[ \quad \gamma(u) := \Gamma(u+1)$$

Then we have with (2):

$$\forall v \in ]-1;\infty[ \quad \gamma(v+1) = (v+1)\,\gamma(v) \tag{7}$$

In addition we have with (6):

$$\gamma \mid [1;\infty[ \text{ is monotonically increasing} \tag{8}$$

Further we define for all $\alpha \in \mathbb{R}_+$ the mapping $f_\alpha : \mathbb{R}_+ \to \mathbb{R}$ through

$$f_\alpha := \sum_{n=0}^{\infty} \frac{(-1)^n}{\gamma(n+\alpha)} \cdot x^{n+\alpha} = x^\alpha \cdot \left( \sum_{n=0}^{\infty} \frac{(-1)^n}{\gamma(n+\alpha)} \cdot x^n \right) \tag{9}$$

We have (because of (3) and (8)) for all $\alpha \in \mathbb{R}_+$, $t \in \mathbb{R}$ and $k \in \mathbb{N}$ mit $k \geq 1$:

$$\begin{aligned}
\sum_{n=0}^{k} \left| \frac{(-1)^n}{\gamma(n+\alpha)} \cdot t^n \right| &= \frac{1}{\gamma(\alpha)} + \sum_{n=1}^{k} \frac{1}{\gamma(n+\alpha)} \cdot |t|^n \leq \\
&\leq \frac{1}{\gamma(\alpha)} + \sum_{n=1}^{k} \frac{1}{\gamma(n)} \cdot |t|^n = \\
&= \frac{1}{\Gamma(\alpha+1)} + \sum_{n=1}^{k} \frac{1}{\Gamma(n+1)} \cdot |t|^n = \\
&= \frac{1}{\Gamma(\alpha+1)} + \sum_{n=1}^{k} \frac{1}{n!} \cdot |t|^n \leq \\
&\leq \frac{1}{\Gamma(\alpha+1)} + \sum_{n=0}^{k} \frac{1}{n!} \cdot |t|^n \leq \\
&\leq \frac{1}{\Gamma(\alpha+1)} + e^{|t|}
\end{aligned}$$

So (9) defines a differentiable mapping and because of (2), (7) and (9) we have for all $\alpha \in \mathbb{R}_+$ :

$$
\begin{aligned}
\left(f_\alpha\right)' &= \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{\gamma(n+\alpha)} \cdot x^{n+\alpha}\right)' = \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{\gamma(n+\alpha)} \cdot \left(x^{n+\alpha}\right)' = \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{(n+\alpha)}{\gamma(n+\alpha)} \cdot x^{n+\alpha-1} = \\
&= \frac{\alpha}{\gamma(\alpha)} \cdot x^{\alpha-1} + \sum_{n=1}^{\infty} (-1)^n \frac{(n+\alpha)}{\gamma(n+\alpha)} \cdot x^{n+\alpha-1} = \\
&= \frac{\alpha}{\gamma(\alpha)} \cdot x^{\alpha-1} + \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(n+\alpha+1)}{\gamma(n+\alpha+1)} \cdot x^{n+\alpha} = \\
&= \frac{\alpha}{\gamma(\alpha)} \cdot x^{\alpha-1} - \sum_{n=0}^{\infty} \frac{(-1)^n}{\gamma(n+\alpha)} \cdot x^{n+\alpha} = \\
&= \frac{\alpha}{\Gamma(\alpha+1)} \cdot x^{\alpha-1} - f_\alpha = \\
&= \frac{1}{\Gamma(\alpha)} \cdot x^{\alpha-1} - f_\alpha
\end{aligned}
$$

Now we have proved:

$$
\forall \alpha \in \mathbb{R}_+ \quad \begin{pmatrix} f_\alpha \text{ is differentiable and} \\ \text{it suffices the ordinary} \\ \text{linear differential equation} \\ \left(y_\alpha\right)' + y_\alpha = \dfrac{1}{\Gamma(\alpha)} \cdot x^{\alpha-1} \text{ on } \mathbb{R}_+ \end{pmatrix} \qquad (10)
$$

# 4. Solution of the ODE

In [2] one can find the following theorem:

**Theo.:**

**Pre.:** Let $J$ be a non-emtpy open interval of $\mathbb{R}$.
Let $g : J \to \mathbb{R}$ be a continuous mapping.
Let $h : J \to \mathbb{R}$ be a continuous mapping.
Let $\xi \in J$.
Let $\eta \in \mathbb{R}$.

**Ass.:** The initial-value problem

$$y' + g(t)\, y = h(t) \qquad y(\xi) = \eta \qquad t \in J \tag{11}$$

has exactly one solution. It exists in all of $J$.

**Rem.:** Let $G : J \to \mathbb{R}$ be the antiderivative of $g : J \to \mathbb{R}$ with $G(\xi) = 0$, i. e.

$$\forall t \in J \qquad G(t) = \int_{\xi}^{t} g(\tau)\, d\tau$$

Then the solution of the initial-value problem above is:

$$\forall t \in J \qquad y(t) = e^{-G(t)} \cdot \left( \eta + \int_{\xi}^{t} h(\tau) \cdot e^{G(\tau)}\, d\tau \right) \tag{12}$$

# 5. Application of the Previous Theorem

Let $\alpha \in \mathbb{R}_+$.
Let $x = \left(\mathrm{id}_{\mathbb{R}}\right) \mid \mathbb{R}_+$.
In the specific case of section 3. is $J = \mathbb{R}_+$ and the mappings $g : J \to \mathbb{R}$ and $h : J \to \mathbb{R}$ are defined by

$$\forall t \in J \quad g(t) := 1$$

$$\forall t \in J \quad h(t) := \frac{1}{\Gamma(\alpha)} \cdot t^{\alpha - 1}$$

We now define a mapping $T_\alpha : \mathbb{R}_+ \to \mathbb{R}$ by

$$\forall t \in J \quad T_\alpha(t) := \Gamma(\alpha) \cdot f_\alpha(t) \cdot e^t \tag{13}$$

We prove finally:

$T_\alpha$ is a antiderivative of $x^{\alpha-1} \cdot e^x$ on $\mathbb{R}_+$

i. e.

$$\forall t, \xi \in J \quad T_\alpha(t) - T_\alpha(\xi) = \int_\xi^t \tau^{\alpha-1} \cdot e^\tau \, d\tau \tag{14}$$

Proof of (14):
Let $\xi \in J$. The antiderivative $G : J \to \mathbb{R}$ of $g : J \to \mathbb{R}$ with $G(\xi) = 0$ has the following form:

$$\forall t \in J \quad G(t) = \int_{\xi}^{t} g(\tau)\, d\tau = t - \xi$$

Because of (10), $f_\alpha$ is a solution of the initial-value problem

$$y' + g(t)\, y = h(t) \qquad y(\xi) = f_\alpha(\xi) \qquad t \in J$$

it follows with (12) for any $t \in \mathbb{R}_+$:

$$f_\alpha(t) = e^{\xi - t} \cdot \left( f_\alpha(\xi) + \int_{\xi}^{t} h(\tau) \cdot e^{\tau - \xi}\, d\tau \right) =$$

$$= e^{-t} \cdot \left( f_\alpha(\xi) \cdot e^{\xi} + \int_{\xi}^{t} h(\tau) \cdot e^{\tau}\, d\tau \right) =$$

$$= e^{-t} \cdot \left( f_\alpha(\xi) \cdot e^{\xi} + \frac{1}{\Gamma(\alpha)} \int_{\xi}^{t} \tau^{\alpha - 1} \cdot e^{\tau}\, d\tau \right)$$

This can be transformed:

$$\forall t \in J \quad \Gamma(\alpha) \cdot \left( f_\alpha(t) \cdot e^{t} - f_\alpha(\xi) \cdot e^{\xi} \right) = \int_{\xi}^{t} \tau^{\alpha - 1} \cdot e^{\tau}\, d\tau$$

i. e.

$$\forall t \in J \quad T_\alpha(t) - T_\alpha(\xi) = \int_{\xi}^{t} \tau^{\alpha - 1} \cdot e^{\tau}\, d\tau$$

So (14) is proved.

# 6. Limites

Let $\alpha \in \mathbb{R}_+$

Let $x = (\mathrm{id}_{\mathbb{R}}) \mid \mathbb{R}_+$.

With (9) we have:

$f_\alpha : \mathbb{R}_+ \to \mathbb{R}$ is continuously extendible in 0 and

$$\lim_{\xi \to 0+} f_\alpha(\xi) = 0$$

With (13) we have:

$T_\alpha : \mathbb{R}_+ \to \mathbb{R}$ is continuously extendible in 0 and

$$\lim_{\xi \to 0+} T_\alpha(\xi) = 0$$

It follows with (14):

$$\forall t \in \mathbb{R}_+ \quad \int_\xi^t \tau^{\alpha-1} \cdot e^{\tau} \, d\tau \text{ converges for } (\xi \to 0+)$$

and

$$\forall t \in \mathbb{R}_+ \quad T_\alpha(t) = \lim_{\xi \to 0+} \int_\xi^t \tau^{\alpha-1} \cdot e^{-\tau} \, d\tau = \int_0^t \tau^{\alpha-1} \cdot e^{-\tau} \, d\tau$$

and

You can get an antiderivative of $x^{\alpha-1} \cdot e^{\beta x}$ by the substitution $\tau \mapsto \beta\tau \; (\beta \in \mathbb{R}_+)$.

# 7. Literature


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